Martin Sætre
Martin.Saetre(at)uib.no

Wittgenstein on Mathematical Symbolism: A Response to Stenlund's Historical Interpretation

Abstract

In recent work, Sören Stenlund (2015) contextualizes Wittgenstein's philosophy of mathematics as being in alignment with the tradition of symbolic mathematics. In the early modern era, mathematicians began using formal methods disconnected from any obvious empirical applications, transforming their subject into a purely symbolic discipline. With this, Stenlund argues, they were freeing themselves from ancient ontological presuppositions and discovering the ultimately autonomous nature of mathematical symbolism, which eventually inspired Wittgenstein's thinking. On this interpretation, Wittgenstein held that mathematical symbolisms are logically isolated and understood independently of their domains of empirical application. This paper examines this narrative and concludes that Stenlund's view of mathematical progress contrasts with the later Wittgenstein's writings, which emphasize grammatical and practical links between mathematics and its areas of application.

In a recent article, Sören Stenlund (2015) combines a philosophical, historical, and exegetical approach in an effort to historicize Wittgenstein's writings on mathematical symbolism. In Stenlund's view, ancient mathematics was characterized by philosophical beliefs rooted in the concept of '*arithmos*', the precursor to the modern concept of number. Ancient mathematics rested on ontological presuppositions, as numbers were taken to be inextricably tied to physical or ideal magnitudes. As mathematics advanced in the early modern era, mathematicians began to free their subject from these philosophical constraints. Through symbolic and methodological innovations, mathematics underwent an essential change: from a science of quantity it was transformed into a purely autonomous symbolic discipline.

For Stenlund, Wittgenstein regarded the freestanding symbolisms resulting from these innovations as constituting the only authentic mathematics in our time, and saw stubborn adherence to the antiquated ontological model of mathematics as an engine of philosophical confusion. This is a compelling picture, but I shall argue that Wittgenstein's remarks on symbolism and its relation to language point in a more *practical*, less theoretical direction than what this picture suggests. I focus on the structure of Stenlund's (2015) historical argument insofar as it serves to attribute a certain model of philosophical progress to Wittgenstein. Most issues of mathematical detail are ignored; the interested reader is invited to consult the referenced articles. The topic here is the extent to which Wittgenstein can be said to share its general understanding of the history of mathematics as tied to philosophical progress.

The paper proceeds by outlining Stenlund's (2015) descriptions of respectively ontological and symbolic mathematics, along with the characterization of the transition from the former to the latter. Then, the notion of 'symbolism' is brought into focus, and it is shown that Stenlund projects a historical understanding of 'mathematical prose' onto Wittgenstein's remarks. While this historicization amounts to an interesting and pertinent perspective on Wittgenstein's writings on mathematics, it does not fully reflect the philosophical implications of these writings. Although Wittgenstein should be contextualized and brought into debates over the history of mathematics, I argue that he would have rejected Stenlund's understanding of the philosophical implications of the invention of symbolic mathematics.

1. 'Arithmos' *and ontological mathematics*

For his historical account, Stenlund (2015) draws primarily on a classic study of mathematical history by Jacob Klein.[1] In essence, Klein (1936/1968) argued that the invention of modern algebra and the various related mathematical disciplines represented, not a gradual and linear change marked by conceptual continuities with earlier methods of calculation, but a fundamental transformation of the entire subject of mathematics. Pivotal to this line of thought is the idea that the contemporary concept of 'number' is logically distinct from the concept of '*arithmos*' as employed by the ancient Greeks. This latter concept would be more correctly translated as "quantity", i.e. number of things, and its use was conceptually inseparable from acts of

[1] Also, to a lesser extent, Unguru (1975, 1991, 1994).

counting-off (Klein 1936/1968: 46). The ancients, in Klein's view, simply did not have the abstract concept of 'number' that is used in modern mathematics.

Klein surveys ancient writings, including Plato and Aristotle, to make his case that '*arithmos*' was seen as a genus with subspecies divided into the things of which the quantities were said to be. For example, ten dogs are distinct from ten apples, and correspondingly different subspecies of 'ten' are counted. The limiting case is the use of "*arithmos*" in reference to mere 'units', but even here the term is connected to counting 'in the abstract'. As Klein (ibid.: 48) says, "this means that a number is always and indissolubly related to that of which it is the number". This is not to suggest that the ancients operated with an empiricist conception of number, since, again, the genera of *arithmos* included immaterial 'units', with subspecies including counts of 'two units', 'three units', etc. On the other hand, it does follow that 'one', like 'zero', was not a number, since "one unit" was effectively pleonastic.

In the ancient Greek context Klein surveys, the use of 'number' was essentially representational, with "*arithmos*" apparently being inseparable from the application of counting nouns. Stenlund (2015, p. 16) elaborates on this with the idea that pre-modern mathematics was an 'ontological' pursuit. Philosophy and mathematics, he says, were intertwined and could not be separated out as their own subjects in the way in which we are accustomed. While there is historical truth to this claim, considering that academic specialization is a relatively recent phenomenon, it is more tenuous to generalize from what we have of philosophical and mathematical sources to draw the conclusion that ancient mathematics as such, and/or the ancient (analogue of our) concept of 'number', was ontologically charged on a conceptual level.

The later Wittgenstein described language games to bring out the practical ramifications of concepts such as 'number' and 'calculation' (e.g. *PI* 7, *RFM* I, 143). This method might help clarify some of the broader differences between our own concept of 'number' and the purportedly archaic conception of '*arithmos*'. First, we should get a better idea of the distinction. As it stands, the archaic conception seems unfamiliar not only because it appears to require a link between numbers and procedures of counting objects, but because it implies an internal relation between numbers and the collections the size of which those numbers are used to measure. While we operate with numbers using both nouns ("four plus 19 make 23") and adjectives ("you have three apples, I have four; that gives us seven apples"), it would appear that the archaic concept strictly admitted constructions of the latter form.

If we take a closer look at what Klein suggests about the use of "*arithmos*", we see that number words were not used with a familiar adjectival or quantificational function at all. Rather, numbers were part of the identification of things, like suffixes or indices that differentiate multiple bearers of a name. To use Aristotle's example (cf. Klein, 1936/1968: 48), "ten" and "nine" qualify dogs in the same way that "scalene" and "equilateral" qualify triangles. On this conception, 'ten dogs' is not merely a different quantity than 'nine dogs', it is a different *thing* than 'nine dogs' altogether; *dogs* as such are seen as a single genus, but the numbers 'nine' and 'ten' mark a contrast in species between odd and even numbers and a distinction in subspecies, their quantity. In contrast, when *we* use expressions such as "number of dogs", "liters of milk", "kilograms of fruit", "hours of work", etc., we speak of potentially fluctuating quantities, using adjectives or quantifiers ranging over given *classes* of things that can vary in number.

To attempt to illustrate, formulating a language game in which the archaic conception *might* seem less perplexing and not just as the result of philosophers imposing theoretical constraints on numbers, imagine a people who do not use currency for trade. When trading, say, apples for oranges, these imaginary people do not price goods by reference to their rates of exchange into currency, but instead compare items directly. Someone with five apples finds someone with oranges willing to barter. That is to say, they agree on trading a *multitude* of apples for a multitude of oranges. To describe this in terms of "sets" would mislead us, since *we* would automatically evaluate and compare the respective cardinalities of each set. A person in this language game trades something, namely five apples, for *something* else, say six oranges, without thereby dividing either into elements, 'set members', of proportional value. It can be assumed that an exchange would only have been made with *exactly* these five apples.

To give an indication of possible consequences for arithmetic, if the apple seller engages in other trades, selling various numbers of apples, these actions differ in character, not just in quantity, from the trade just described. Subtracting three apples would be a *distinct operation* from subtracting five apples, although they might be related, similarly to how the construction of a scalene and an equilateral triangle are related but distinct procedures. Here, one might speak of homogeneity and proportionality in instances of apples-to-apples calculation, but doing so in instances of apples-to-oranges calculation might reasonably be seen as confusion.

In such a language game, numbers could be said to function nominally or indexically, specifying *what* is offered in a trade as opposed to quantifying classes of commodities. In other words, the use of "number" would here be more along the lines of how Klein and Stenlund describe '*arithmos*'. Naturally, someone might respond to this proposal by questioning whether a language game involving barter could determine the concept of 'number' in an entire culture. However, the point here was merely to sketch an example of the kind of features a form of life might have as *part of* a different approach to mathematics, and thereby to present one way of understanding – from a perspective in line with the later Wittgenstein – the contention that "number" had a different meaning in a hypothetical premodern context.

2. The emergence of symbolic mathematics

The above is by no means intended as an historically accurate description of the context behind e.g. Aristotle's use of "*arithmos*", nor an endorsement of the idea that numbers were understood in this way in any given ancient society. It bears repeating that the extent to which theoretical historical writings are good indicators of concepts, as opposed to more or less idiosyncratic perspectives or constructs, is an open question. Of course, it is sometimes all we have to go on.

Rather, it is meant as a sketch of a philosophical *approach* to understanding a (purportedly) premodern concept of 'number' that is more practically oriented than Stenlund's (2015, 2014) own way of characterizing 'ontological mathematics', which focuses on the results and innovations of individual mathematicians and philosophers. The above shows that an approach to understanding alternative concepts of 'number' taking its cue from the later Wittgenstein's method of language games has the potential to throws such concepts into a different, less theoretical, light. With that in mind, we turn to the emergence of symbolic mathematics, which marked a significant shift in the history of the subject.

The development of symbolic mathematics can be traced to the principle of positional numeration, which was invented by Arabian mathematicians and carried over into medieval Europe.[2] This principle characterizes number systems that distinguish the meaning of numerals on the basis of their relative position in an expression; we multiply the '2' in '120' by 10, and the '2' in '200' by 100. Symbolic mathematics took off around the 17th century with the generalization of arithmetical functions and the invention of symbolic algebra. In Klein and Stenlund's account, this involved the disassociation of 'number' from quantities of definite things, with 'number' becoming a concept of its own rather than a genus of objects.

Stenlund (2014, 2015) describes the thrust of this transformation as a change from the ancient, ontological conception of numbers as abstract entities, to mathematics becoming a purely symbolic discipline.[3] He highlights Franciscus Vieta's '*logistica speciose*', a form of algebra characterized by the systematic use of letters in place of numerals, with unknown magnitudes represented by vowels and given magnitudes by consonants. The general nature of this notational innovation freed mathematicians from having to stipulate idiosyncratic rules in order to solve specific numerical equations. The "*speciose*" refers to Vieta's understanding of algebraic expressions as distinct *species*, apparently not dissimilar from the ancient Greek conception of 'number', but Stenlund (2015: 17–18) quotes Klein as to the fundamental difference that Vieta's symbolic approach made to this conception:

> The species are in themselves symbolic formations – […] They are, therefore, comprehensible only within the language of symbolic formalism. […] Therewith the most important tool of mathematical natural science, the "*formula*", first becomes possible, but above all, a new way of "understanding," inaccessible to ancient *episteme* is thus opened up. (Klein, 1936/1968: 175)

Rather than being given meaning through an association with concrete quantities or units of measurement, numbers were now available to be understood purely formally, as *constituted* by the 'species of expression' or the symbols involved in algebraic operations. It is this notion of 'mathematical

[2] See Dantzig (1930/2005: 33).

[3] It might be questioned whether Stenlund differs from Klein in his understanding of (the symbolic concept of) 'number'. For example, Klein (1968: 193–194) writes that, with Stevin's realization of the unlimited possibility of forming ciphers, "the symbolic understanding makes 'number,' [...] appear as a 'material' comparable to the material of bread or water." This exegetical question will not be pursued here.

symbolism' as a purely formal and non-referential system of signs that Stenlund (2015: 25) contends is both highlighted and championed by Wittgenstein.

Stenlund distinguishes the notion of a 'notation', a set of signs considered in isolation from their use, and 'symbolism'. A symbolism is determined by symbolic *operations*, with each symbol in a symbolism being defined through one or more operations involving other symbols. Stenlund (2015: 23) writes that a symbolism is "not just a system of notation in the typographical or linguistic sense", but while the inclusion of operations certainly distinguishes it from a typographic system, the general allusion to *linguistics* is less clear, considering that a syntactic system of signs would typically include structural rules. In any case, the dichotomy between symbolism and notation resonates with Wittgenstein's writings, with probably the most obvious parallel, and the one drawn by Stenlund (2015: 58–59, 70), being the distinction between 'sign' and 'symbol' in *Tractatus Logico-Philosophicus*.

For the early Wittgenstein (*TLP*: 3.318–3.326), a sign is a mark of an expression, a visible or audible pattern, while a symbol is a sign together with (one of) its logico-syntactic *use*(*s*). The sentences "Alice is old" and "76 is more than 75" contain the same sign, "is", while expressing two different symbols. Crucially, the difference is understood in terms of linguistic practice rather than reference. Someone *could* extrapolate from this distinction and develop a concept of 'symbolism' as an arbitrary set of signs along with ways those signs are used. The meaning of the symbols would then be identified with their role in the symbolism, irrespective of the application of the symbolism itself and how the signs function in other parts of language. It is doubtful that Wittgenstein ever had such an idea in mind, however, even in his early period.

Wittgenstein goes on to use "symbolism" and "calculi" to refer to mathematical systems, for instance in his discussions of the system of *Principia Mathematica* (e.g. *RFM,* I, App. I: § 6). Here, although he rejects the idea that mathematical symbols refer to entities and that formulae are empirical propositions, stressing logical autonomy in *that* sense, he at the same time emphasizes grammatical and practical connections between pure mathematics and linguistic practices more broadly. For Stenlund (2015: 46), however, the emergence of symbolic mathematics involved a separation of pure mathematics from its applications: "[A]n essential feature of the symbolic point of view was the logical separation of a symbolic system from its application to some subject-matter outside pure mathematics."

In the remaining sections of this article I will argue against this conception of symbolic mathematics as essentially pure mathematics. I highlight that, for Wittgenstein, mathematics hangs together with grammar, implying that we should not think of mathematical symbolisms as 'logically separated' from any domain of linguistic practice in which we might apply them. This reflects the later Wittgenstein's view that the function of signs as used in a symbolism (or a "sign-game") has to be related to how they are used *outside* of mathematics (*RFM*, V: § 2). The argument still leaves intact the notion that symbolisms may be *referentially* and *representationally* autonomous, but it nevertheless implies that the historical development of mathematics should be seen in connection with broader changes in linguistic practices.

3. Mathematical logic and historicizing prose

Stenlund argues that Wittgenstein criticized the tendency to explain and describe the formulae and techniques of modern mathematics using verbal language, for example when teaching and discussing advanced mathematics in a university setting. This tendency assumes (erroneously, as far as Stenlund is concerned) that the symbols used in mathematical symbolisms can be translated into the vernacular without introducing distortions:

> [W]hat is still not abandoned is the tendency to give meaning and significance to basic notions in mathematics and formal logic by translation or paraphrase into verbal language (to which I count what is often called "informal mathematical language", or, in Wittgenstein's words "mathematical prose".) (Stenlund, 2015: 37)

One noteworthy aspect of Stenlund's argument is its historical framing. The timeline involved can be illustrated with Nesselmann's (1969/1842) simplified division of the history of algebra into three expressive stages: the ancient, rhetorical stage, featuring full sentences describing manipulations of objects, followed by the syncopated stage, featuring abbreviations of certain terms and rules for using them, followed by the modern, fully symbolic stage.[4] In terms of that model, Stenlund appears to interpret Wittgenstein's allusions to mathematical prose as critiquing relapses into rhetorical or syncopated forms

[4] Nesselmann's tripartite model of algebra has serious difficulties, besides being an oversimplification, as argued by Heefer (2009: 1–4). The criticisms are relevant for the uniqueness of 'symbolic mathematics', but they require a longer discussion. The model is referenced here to set the stage for Stenlund's (2015) historicization of 'prose', building on Klein (1936/1968, which touches on Nesselmann's work).

of expression. Agreeing with Klein (1936/1968) that the symbolic stage effected an essential transformation, Stenlund regards this charge as devastating, involving a confusion of mathematics with an ontological enterprise.

There is little in terms of a sustained investigation of the history of mathematics in Wittgenstein's writings, making is difficult to know precisely how he sees the development of modern mathematics and its relation to ancient mathematical practices. As will be shown, he uses the notion of 'mathematical prose' to criticize *distortive* attempts to translate mathematics into verbal language, but it does not follow from this that he rejects the widespread practices of paraphrasing mathematics into ordinary language in and of itself.[5] Moreover, it is not clear that Wittgenstein conceives of mathematical prose as a historical relic, at least not in the way that Stenlund suggests by lamenting that the tendency in question is "still not abandoned".

As Stenlund sees it, there is a tendency to articulate "modern mathematics in ordinary verbal language by assigning a place for mathematical propositions in the general category of propositions expressed by declarative sentences of natural language", (2015: 35) and reads Wittgenstein as criticizing this tendency.[6] An example of this is his interpretation of *RFM*, V: § 46. In this remark, Wittgenstein writes about "the curse of the invasion of mathematics by mathematical logic", a 'curse' involving the translatability of any proposition into a formal logical symbolism in a way in which potentially meaningful differences are obscured.

Wittgenstein's worry here is that applying mathematical logic introduces vagueness masquerading as clarity. Mathematical logic, he adds, "makes us feel obliged to understand it. Although of course this method of writing is nothing but the translation of vague ordinary prose" (*RFM*, V: § 46). As an example, the two different sentences "at least one natural number has the property F" and "the set F has a member" might both be formalized as $\exists x(Fx)$. In this case, the 'vague ordinary prose' is a sentence such as "there exists an x such that F of x", the formalization paving over differences between sentences that are expressed in verbal language.

[5] The notion of a 'paraphrase' is arguably too strong; in teaching and conveying mathematics, an extended form of natural language sometimes called "mathematese" is typically employed.

[6] Note that, for the middle and later Wittgenstein, there *is* no general category of propositions.

From this remark, Stenlund (2015, p. 57) infers that "one important example of 'prose accompanying the calculus' is the ordinary language expressions used in the translation of the signs and formulas of the predicate calculus into verbal language". However, as we just saw, Wittgenstein's critical remarks in and surrounding *RFM*, V-46 ostensibly do not just target the translation of the predicate calculus into language, but the generality of the predicate calculus itself. That is, he targets the idea that mathematical logic is the way to properly understand mathematical formulae when, in reality, it effectively fares no better than vague prose.

Another noteworthy aspect of *RFM*, V: § 46 is that Wittgenstein here leaves us a clue with respect to historical framing. According to Stenlund (2015: 37), our tendency to paraphrase mathematics into verbal language is a relic of a distant past in which ontological considerations played a role in mathematical practice. In ancient Greece, mathematics was held to reflect the most general features of reality, features which remained after all contingent properties and relations had been "abstracted from" material objects. The nature of, and access to, *abstracta* was thus a major intellectual concern, as illustrated by the writings of Aristotle. By requiring that mathematics be translatable into logical forms familiar from ordinary language, the ancients sought to remain faithful to a preconceived ontological reality, constraining what was deemed mathematically permissible. Thus, mathematical prose, for Stenlund (2015: 34–37, 40) has its source in what he calls "the Euclidean-Aristotelian heritage", and progress towards a symbolic framework in mathematics has come together with philosophical clarity.

The remark *RFM*, V: § 46 which Stenlund (2015: 57) cites, however, indicates a contrasting timeline. Wittgenstein here writes that mathematical logic, deriving from the 19th century, has the effect that "*now* any proposition can be represented in a mathematical symbolism," my emphasis. The topic Wittgenstein deals with here is ostensibly not an artefact of a bygone era, but a live, even relatively *recent*, tendency in mathematics and philosophy. Similar points can be made for most other critical remarks Wittgenstein makes on mathematics. Even his reference to "mathematical alchemy" in *RFM*, V: § 16 – which at least alludes to a premodern phenomenon – is explicitly in response to a *modern* tendency in mathematics: "[T]he whole system of pretense […] that by using the new apparatus we deal with infinite sets with the same certainty as hitherto we had in dealing with finite ones" (*RFM*, V: § 15).

This is not to deny Stenlund's historical observations insofar as they pertain to ancient philosophical conceptions of mathematics; Wittgenstein undoubtedly criticizes ontological – or extensionalist and representationalist – presuppositions about mathematics that can readily be *compared* to views that we might find in, for example, the writings of Plato and Aristotle. However, even if ontological conceptions of mathematics first gained prominence around the time of Plato, Aristotle, and Euclid, it does not follow that Wittgenstein rejected ontological conceptions *by way of contrast* with modern mathematics, taken as philosophically-speaking relatively unproblematic. By all accounts Wittgenstein felt that such conceptions were alive in his time, and arguably that they were growing rather than abating in prominence.[7]

Wittgenstein's criticism of the foundational role given to mathematical logic, as involving an ontological misunderstanding, extends to the analysis of language. He writes that a concept is not essentially a predicate (*RFM*, V: § 47), commenting on both Aristotelian logic as well as modern mathematical logic, mentioning that the latter builds on the former (*RFM*, V: §§ 40, 48). The problem Wittgenstein sees in connection with mathematical logic is that of an unwarranted sense of explanation stemming from the mere possession of a general method of translation. This distorts our understanding of both mathematics *and* ordinary language:

> 'Mathematical logic' has completely deformed the thinking of mathematicians and of philosophers, by setting up a superficial interpretation of the forms of our everyday language as an analysis of the structures of facts. (*RFM*, V: § 48)

In other words, the problem for Wittgenstein is not that we draw on verbal language whenever we paraphrase mathematical formulae, and that this translation into verbal language gives an ontological gloss on mathematics. Rather, the problem is that formal logic imposes a superficial, extremely general, interpretation of grammar – in *this* way building on Aristotelian logic – and yet the process of translating formulae or sentences into formal logic is treated as if it were a general method of explaining the *structure* of mathematics or language.

[7] This is not to suggest that Stenlund attributes the specific historical view that mathematics transitioned from ontological science to symbolic discipline (Klein 1968: 184) to Wittgenstein. What I am contesting is the notion that the later Wittgenstein saw mathematical prose as a 'thing of the past', an antiquated tendency in the way that Stenlund (2015: 37, 56) suggests, and that Wittgenstein contrasted confused conceptions of mathematics with symbolic mathematics, taking the latter to be the most "authentic" to modern times (2015: 35).

Stenlund addresses subfields of mathematics which he takes to deviate from the overall thrust of philosophical progress initiated by modern mathematics. He discusses the "ontological mythologies of transfinite set theory" (2015: 35) and logical semantics, while framing these developments as "cementing the old ontological view of mathematics" (2015: 55). In discussing these subfields he provides much needed context to many of Wittgenstein's remarks, helping explain the critical nature of (especially the middle) Wittgenstein's writings, while drawing illuminating links to the perspectives of prominent mathematicians. Wittgenstein's comparison of mathematics to chess figure prominently in Stenlund's reading, and he links this comparison to writings of mathematicians such as Thomae and Couturat.

According to Stenlund (2015: 50), "The arithmetical calculus, like the game of chess, is autonomous". Now, while Wittgenstein does call mathematics a "family of games", it should be kept in mind that he also contrasts mathematics with "mere sign-games" (*RFM*, V: § 2, *RFM*, VII: § 33). He sees mathematics as embedded in human practices, with mathematical techniques being seamlessly integrated into various activities.[8] Moreover, in the middle and later periods, Wittgenstein prominently compares *language* to chess, such as in *PI*: §41. Hence, the analogy between mathematics and chess does not by itself support the idea of a 'distance' between mathematics and language. His allusions to chess generally focus on the idea that chess-pieces (like words or symbols) are constituted by their *use* in the game, the point being that language and mathematics are *activities*: "The word 'language-game' is used here to emphasize the fact that the *speaking* of language is part of an activity, or of a form of life" (*PI*: §23).

By what can be gathered from this remark, the point of appealing to games like chess is not to highlight the *autonomy* of language or mathematics, but to highlight their *practical* nature. Regardless of their arbitrariness, games are only 'autonomous' in a very limited sense. No game entails the way it is played, since rules do not determine their own following. Games have indefinitely many implicit features that are not explicitly decreed, which must be taken for

[8] See also *LFM*, XV: 142: "The thing to do is not to take sides, but to investigate. It is sometimes useful to compare mathematics to a game and sometimes misleading," along with his elaboration of the chess-analogy in *LFM*, XV: 143–144. Chess is unlike mathematics in having no obvious application. That is not to say that practical utility is a necessary condition for all forms of mathematics; see *RFM*, I: § 167.

granted in practice, such as the fact that players *desire* to fulfil their win-condition(s).[9]

Stenlund understands symbolisms not just as sets of signs, but as systems of symbols with a meaning *in* the symbolism, similarly to how chess-pieces have defined roles in the game. However, his stress on the self-contained nature of chess *overall*, and hence of mathematical symbolisms, implies a variety of strong formalism (cf. Nakano 2020). For present purposes, 'strong formalism' can be defined as the view that a symbolism is conceived and used in isolation, ensconced from all practical and social context, independently of any external applications or point(s) it might possess. Crucially, for strong formalism, there need not be any relevance to how the signs of a symbolism are used *outside* that symbolism, this being a contingent relation, which contrasts with what Wittgenstein says in *RFM*, V: § 2.

Stenlund (2015: 26) distances himself from "superficial" varieties of formalism which deprive mathematical signs of any meaning. Still, by underlining the autonomy of chess as key to the comparison with mathematics, he appears to cut this meaning off from outside influence. Agreeing with Thomae, Stenlund writes that the signs of arithmetic "have a content determined by the forms of their use – not by possible applications of the system" (2015: 51). This formalist view has ramifications for his historiography, leading to a strong focus on pure mathematics and a severing of any link between the emergence of modern mathematics, understood purely theoretically, and the changing circumstances involved in its practical applications.

However, it is difficult to see how Wittgenstein's later writings fit with the notion that the general application of a symbolism is irrelevant to the 'content' of its signs. He highlights the importance of the *point* of a game in *RFM*, I, App. I: § 20 and *LFM*, XXI: 205. A game like chess takes on an entirely different guise depending on the circumstances, such as whether a draw is deemed acceptable. In several qualifications and modifications of the analogy, Wittgenstein argues that chess would be akin to mathematics *had it been* systematically incorporated into serious human decision-making (*WWK*: 163, 170; *LFM*, XV: 143).

[9] Wittgenstein makes a similar point in *PI*: § 68, with the tennis-example. On his use of the chess-analogy, and how it might be misunderstood in ways that are relevant here, see Gustafsson (2019) and Conant (2019).

In any case, the main point in this section concerns the historicization of Wittgenstein's views. Given that Wittgenstein criticizes the use and motivations of important subfields of modern mathematics, such as axiomatic set theory, it is questionable to attribute to him a *philosophical* preference for modern mathematics over older forms of mathematics. One reason for hesitation is that such a claim requires an empirically disputable timeline of the emergence of various mathematical subfields, which means that developing and defending such a historiography would too easily devolve into a 'No true Scotsman'-style of argumentation. Perhaps more importantly, though, such a framing is at risk of understating the critical and, from a modern perspective, 'radical' nature of Wittgenstein's philosophy of mathematics. Rather than mathematical progress being weighed down by outdated philosophical dogmas, he might just as well say that what is counted as progress in certain subfields of mathematics has come at the cost of new, or reinvigorated, confusions over foundations and applicability.

That being said, it could still be argued that the forms and methods of symbolic mathematics are uniquely congruent with Wittgenstein's philosophy of mathematics, and that symbolic mathematics *does* form the core of contemporary mathematics, even if there are also deviations from it. This would imply that there is a philosophically relevant *gap* between symbolic mathematics and other forms of mathematics. In the final two sections I will argue that, for Wittgenstein, there is no such philosophically relevant chasm between symbolic mathematics and rhetorical or syncopated forms of mathematics, at least not one involving a fundamental distinction in their relation to ordinary language, and that the idea that there is such a distinction involves a potential confusion over the 'abstract' nature of symbolism.

4. The proximity of mathematics to language

According to Wittgenstein in *PI*: §124, philosophy "leaves mathematics as it is, and no mathematical discovery can advance it". So, generally speaking, it is from the outset most charitable to read him as philosophically neutral towards mathematical results.[10] It would *prima facie* seem that Wittgenstein should be read as admitting, as possible, any historical claims pertaining to the

---

[10] That does not entail that he had no opinions about the motivations *behind* mathematical developments. On the contrary, at one point he describes himself as subjecting "the *interest* of calculations to a test" (*RFM*, II–62).

revolutionary history of the development of "number" (or "*arithmos*") made by historians and sociologists of mathematics, while regarding the mathematical developments involved as philosophically neutral, that is, as neither inherently positive nor negative.

That being said, things are not so simple given that the changes under consideration are not on the scale of individual proofs or results, but involve entirely new ways of doing mathematics. For Stenlund (2015: 46–47) in fact, Vieta's symbolic innovations inaugurated a transformation of the very meaning of 'mathematics', resulting in the field of mathematics that we have today. Even in elementary arithmetic and algebra we are no longer calculating with abbreviations for numerically given referents, but with the very symbols in our symbolisms themselves. Mathematics no longer deals with anything *external* to itself.[11] That being so, *PI*: §124 alone is arguably not enough to show that Wittgenstein should be assumed to have seen the emergence of symbolic mathematics as a philosophically speaking neutral phenomenon.

The question, then, is whether the change should be conceived in the way that Stenlund does. Stenlund (2015: 34, 56) writes that mathematics has historically had "proximity with ordinary verbal language" but argues that Wittgenstein's "strict symbolic view of modern mathematics is diametrically opposed to this feature", rejecting the idea of a close relationship between mathematics and language. Both the sense and the truth of these two statements depend on what parts of mathematics and language we have in mind. Stenlund focuses on the relationship between numerical symbols, in particular numerals, variables, and constants, on the one hand, and the referential meaning of nouns and nominal phrases, on the other. In this regard, it could be argued that premodern mathematics has had a closer proximity to everyday language than modern mathematics. For instance, in medieval algebra, a word for "thing" ("*cosa*") was used as a placeholder for an unknown quantity, later to be replaced by symbols.

As a reading of Wittgenstein, however, this focus seems unmotivated, if not misplaced, in light of his extensive critique of the idea that the meaning of

[11] Note that this is a leap: even if we calculate with symbols (or words, or objects; see *RFM*, V: § 2), the activity as a whole gets meaning from its external uses. For example, solving the equation '$x + y = 10$' gets its meaning (not its correctness) from practices in which we replace its signs with actual numbers (i.e. properties of concepts; see *RFM*, VII: § 42). The generalization involved in using such formulae is teachable and understandable by means of informal language combined with practical examples, as documented by Carraher et al. (2008).

language is generally to be understood on the model of nouns and their denotations (e.g. *PI*: §§ 1–23). More in line with Wittgenstein's approach would be to focus on the relationships between mathematical techniques, on the one hand, and grammatical structures involving non-nominal (e.g. verbal or adjectival) expressions, on the other. For example, the use of two or more variables in an equation might not correspond to any noun, but, when applied, such equations *do* correspond to uses of verbs such as "move" or adjectives and adverbs such as "fixed" and "increase" (cf. *RFM*, II: § 30). Note that the environment, and purpose, behind the development of modern algebra included mercantile bookkeeping practices (Hadden 1994: 93) and technical descriptions of motion (Katz 2006: 194–196). People were no longer using and developing mathematics merely to arrive at fixed quantities, but to find precise ways of tracking fluctuating monetary relationships and effective ways of describing curves, respectively.

With these innovations, rather than being *separated* from language, modern mathematics was brought into contact with new domains of linguistic practice. Seemingly attuned to this, in *RFM*, IV: § 15, Wittgenstein airs the possibility of people having exclusively applied (symbolic!) mathematics, and no theorems or unapplied equations of pure mathematics. As he writes, "For this purpose they make use of a system of co-ordinates, of the equations of curves (*a form of description of actual movement*) and of the technique of calculating in the decimal system," emphasis in the original. Overall, he conceives of mathematical development in terms of historically contingent practices rather than theoretical constraints, which lines up with the illustration of "*arithmos*" as hypothetically part of language games of nonmonetary trade, outlined earlier, and suggests a similar approach to understanding symbolic algebra.

Although the Scientific Revolution crucially gave mathematics a new role in our understanding of nature, this novel employment of mathematics was not the inevitable effect of the adoption of positional numeration with decimals. Rather, it was the historical result of a process of integrating new algebraic methods, coupled with associated forms of speech (involving terms like "variable", "constant", "tendency", "increase", "curve", "parameter", etc.), into observational and experimental traditions. According to Hadden (1994: 71–94), these mathematical methods themselves can be traced to commercial practices, such as banking and double-entry bookkeeping, which had proliferated over the 14th and 15th centuries, eventually having a major role in the transformation out of the feudal system in Europe.[12]

In response to this, it might be granted that changes in *applied* mathematics are connected with changing practical circumstances, but that this misses the point: Wittgenstein stressed the gap between pure mathematics and verbal language in his remarks on *proof*. After all, he described attempts to express a proof in verbal language as "misleading", recommending instead that we fix our attention on the actual calculation going on in the proof (see *PG*: 369–370). Stenlund (2015: 56) takes Wittgenstein's discussions of proof to highlight the distinction between what is *sayable* in language and what is *showable* through symbolism.

However, there is a somewhat different way of interpreting Wittgenstein's remarks on proof which eschews his early distinction between saying and showing. By describing the verbally stated result of a proof as "misleading", a theme that recurs throughout Wittgenstein's middle and later writings, he can be read as stressing the *practical* aspect of proof, *proving* as inventing and performing mathematical procedures, over the explicit, static aspect that is formulated in theorems. Accordingly, mathematical *techniques* gain a particularly important role in Wittgenstein's later writings. At one point, he calls a proof "an instrument of language", determining a form of expression (*RFM*, III: §§ 36–37). The form of expression is not the proven theorem, but its manner of construction, a "track" which is laid down in language (*RFM*, III: § 29). In Wittgenstein's view, the proof serves as a model for a reproducible pattern of inference (*RFM*, III: § 44). Mathematical reasoning is further likened to composing "correct (convincing) transitions" in music (*RFM*, III: § 63, cf. *RFM*, I: § 171), the criteria for which are vague. Proofs function as guides for ways of calculating, that is, for doing and applying mathematics, rather than merely serving to convince us of propositions (*RFM*, IV: § 27).

On this reading, Wittgenstein makes the argument that we can be misled by focusing on the verbal expression of a proof because it leads us to *hypostatize* the result into an independent "thing" or "fact" that we discovered, as if it were

[12] According to Hadden (1994: 80), "If we agree that Klein, for example, has successfully established that abstraction and general magnitude are the outcomes of the transition to a symbolic mathematics, then we must attribute this transition to some practice capable of affecting such a transformation. The irrelevance of *kinds* of magnitudes for their numerical comparison is, we argue, attributable to a practice where such *kinds* are already irrelevant to the entity being counted, that is, *value*." Hadden is here describing monetary value.

the outcome of an experiment. Instead, a proof must be followed if it is to be properly understood. Its function is precisely to demonstrate something that we now can *do*, setting up a criterion for action in mathematics and in language. This is continuous with his view that in order to understand the meaning of a word we should attend to its use (*PI*: § 340). This reading also aligns with the suggestion that the relationship between mathematics and ordinary language should, generally speaking, be conceptualized in terms of (verb/adjectival/adverbal) structure rather than (nominal) content.

5. Wittgenstein's approach to symbolism

What we can gather of Wittgenstein's writings and lectures on hypothetical mathematical practices, for instance from his discussion of the 'wood-sellers' (*RFM*, I: §§ 143–152, cf. *LFM*, XXI: 204), indicates that he held that a period's mathematics has to be seen in light of the language and practices of that period. We should not assume the effects of an inevitable, transhistorical, 'rational' trajectory to the present. In this respect he might have agreed with Unguru's (1975: 68–69) assessment of the Whiggish tendency in the history of mathematics: "As to the goal of these so-called 'historical' studies, it can easily be stated in one sentence: to show how past mathematicians hid their modern ideas and procedures under the ungainly, *gauche*, and embarrassing cloak of antiquated and out-of-fashion ways of expression." Stenlund's efforts to situate Wittgenstein's writings within the literature on the understanding of mathematical history, and *vice versa*, are therefore highly pertinent.

Nevertheless, as has been argued, it does not follow that Wittgenstein took sides in favor of a modern, "symbolic point of view" (Stenlund 2015: 56, 58). The question that might be asked here is: A symbolic point of view *on what*? Stenlund's (2015, 2014) use of "symbolic mathematics" appears to oscillate between two meanings, serving both as a term for a non-referential/non-propositional (set of) practice(s), the kind of calculation with Arabic numerals and letters for variables that began in modern times, and as a label for a philosophy or *Weltanschauung* which favors purely symbolic mathematics. For Wittgenstein, I am suggesting, symbolic algebra, to focus on that core example, is decidedly a *practice*. The role of philosophers should be to adequately understand it, including how it overlaps with other forms of mathematics, how it is taught and communicated in language, and how it is applied in various empirical domains. It is not part of a belief system, to be affirmed or denied.

Even if this is granted, it is still possible that symbolic algebra, as a practice, differs in essential respects from other, older mathematical practices. The invention of symbolic algebra was historically important, and Stenlund can be read as pointing out that, in order to understand contemporary mathematics, which builds on symbolic algebra in key areas like analytic geometry and calculus, we have to adopt a perspective which accommodates non-referential mathematical practices. Wittgenstein would agree that such a perspective is needed, but not because symbolic algebra is fundamentally different from earlier mathematical practices and as such can only be accounted for by adopting a perspective befitting it in particular. At the very least his later writings, in which he describes mathematics as a 'family' of variously overlapping practices (*RFM*, V: §§ 32–33), count against him making such a division.

Wittgenstein understands mathematics in terms of non-representational calculating activities *in general* (*RFM*, I, App. III: § 4, *RFM*, VII: § 31), and would seemingly apply this even to the very earliest archaeological records of what might be called "mathematics", such as lists of Pythagorean triplets (numbers satisfying $a^2 + b^2 = c^2$) inscribed, in a kind of sexagesimal notation, on Babylonian clay tablets dating from around 1800 BC (see Abdulaziz 2010). So, if we are to adopt Wittgenstein's perspective, modern symbolic algebra does not stand out in this respect. Just as the use of symbols does not preclude empirical application, the use of full sentences does not preclude an activity from constituting a calculus, either. Already in the early 1930s Wittgenstein called scheduling with a diary a 'calculus' (*WWK*: 171, 168–169). Verbal and symbolic calculation differ from each other, but that is because the details vary, not because one is representational and the other is not. They are typically woven together: consider the act of planning a shopping excursion, writing a list of groceries, and calculating prices.

In short, rather than fixating on forms of expression in isolation, the later Wittgenstein's approach was to investigate how expressions are *used*, and why they are used in that way. An example of this can be seen in *PI*: §§ 189–190, where he discusses different ways in which an algebraic formula is said to 'determine steps in advance'. Due to its simplicity, the importance of the circumstances of the use of a symbolism comes out plainly from this example.

He concludes that the criteria for how a formula is meant include factors such as "the kind of way we always use it, were taught to use it". This again indicates that he would have favored historical studies of mathematics which incorporate anthropological considerations, touching on what brought people to adopt a given notation and in what settings it is used.

When it comes to empirical application, the signs in an equation of elementary arithmetic serve as measures of discrete quantities, whereas signs of a polynomial equation are used as parameters in a model (see Dieudonné 1998: 20). These applications involve different background conditions, the algebraic formula presupposing units of measurement and hence systematic practices of measuring (cf. *PI*: § 242). Nevertheless, from Wittgenstein's perspective, the equations function similarly. They are both part of calculi, that is, reliable, rule-bound activities (*PI PoP*: §§ 347–350), featuring points of contact with contingent and varying acts of counting and measuring, which is what makes them useful in our lives.

Finally, it should be noted that the mathematical operations in both arithmetic and algebra have analogues in, and links to uses of, ordinary language. It is difficult to see how this could fail to be so without losing our grip on 'the point' of the symbolisms, the reasons why people use them in particular ways. Again, Wittgenstein highlighted precisely the human significance, the practical point, of mathematical symbolisms as something that guides the use and development of the symbolisms (*RFM*, I, App. I: §§ 18–20, cf. *LFM*, XXI: 205).

For example, even though the arithmetical use of the sign "+" should be distinguished from the use of the signs "add", "sum", or even "plus" outside of calculations, we nevertheless do arithmetic with "+" precisely in situations in which we would also use words such as "add", "sum", "plus", and "more" to talk about increases or additions.[13] This *does* contrast with the use of "+" in algebraic functions, for instance "$x + y = 10$", but in such cases the analogue is with *other* grammatical uses of "add" or "plus", such as describing the outcome of two factors as being "added together". Symbols used in advanced mathematics often lack such direct counterparts in ordinary language, but if Wittgenstein is right in taking the *raison d'être* of mathematics to be its

[13] The "+"-sign was initially a ligature for "*et*" (Latin for "and") used in written language, similarly to the ampersand. Alongside "−", its mathematical use derives from mercantile practice, indicating a surplus, first attested in Johannes Widmann's 1489 treatise on arithmetic for merchants (see Cajori 1993: 232–234).

applications, then the techniques used in these higher branches of mathematics should still, on some level, be coordinated with ordinary language.

Ultimately, Stenlund (2015) successfully underscores the importance of the historical dimension in understanding Wittgenstein's philosophy of mathematics. Nevertheless, Stenlund's perspective that mathematical symbolisms are detached from language represents a significant departure from the later Wittgenstein's emphasis on the role of mathematics in practice and his contention that mathematics is continuous with grammar. Taking this into account shows that it is wrongheaded to suggest that Wittgenstein had a philosophically motivated preference for modern symbolic mathematics over earlier forms of mathematics.

References

Abdulaziz, A. A., 2010. "The Plimpton 322 tablet and the Babylonian method of generating Pythagorean triples". *arXiv*:1004.0025.

Cajori, F., 1993/1929. *A History of Mathematical Notations*. Mineola, NY: Dover Publications.

Carraher, D. W., Martinez, M. V., Schliemann, A. D., 2008. "Early algebra and mathematical generalization". *ZDM - Mathematics Education* 40, 3–22.

Conant, J., 2019. "Reply to Gustafsson". In: S. Miguens, ed., *The Logical Alien: Conant and His Critics*. Cambridge, MA: Harvard University Press, 863–947.

Dantzig, T., 1930/2005. *Number: The Language of Science*. J. Mazur ed., New York, NY: Pi Press.

Dieudonné, J., 1998. *Mathematics – The Music of Reason*. H.G. Dales and J.C. Dales trans., Berlin: Springer.

Gustafsson, M., 2019. "Wittgenstein on Using Language and Playing Chess: The Breakdown of an Analogy and Its Consequences". In: S. Miguens, ed., *The Logical Alien: Conant and His Critics*. Cambridge, MA: Harvard University Press, 202–221.

Heeffer, A., 2009. "On the nature and origin of algebraic symbolism". In: B. V., Kerkhove, ed., *New Perspectives on Mathematical Practices: Essays in Philosophy and History of Mathematics*. Singapore: World Scientific Publishing, 1–27.

Katz, V. J., 2007. "Stages in the history of algebra with implications for teaching". *Educational Studies in Mathematics* 66, 185–201.

Klein, J., 1936/1968. *Greek Mathematical Thought and the Origin of Algebra*. Cambridge, MA: MIT Press.

Nakano, A. L., 2020. "Wittgenstein, formalism, and symbolic mathematics". *Kriterion: Journal of Philosophy* 61, 31–53.

Nesselmann, G. H. F., 1969/1842. *Die Algebra der Griechen*. Berlin: Minerva.

Stenlund, S., 2014. "The Origin of Symbolic Mathematics and the End of the Science of Quantity". *Uppsala Philosophical Studies* 59.

Stenlund, S., 2015. "On the Origin of Symbolic Mathematics and Its Significance for Wittgenstein's Thought". *Nordic Wittgenstein Review* 4, 7–92.

Unguru, S., 1975. "On the need to rewrite the history of Greek mathematics". *Archive for History of Exact Sciences* 15, 67–114.

Unguru, S., 1991. "Greek Mathematics and Mathematical Induction". *Physis* 28, 273–289.

Unguru, S., 1994. "Fowling after Induction, Reply to D. Fowler's comments". *Physis* 31, 267– 272.

Wittgenstein, L., 1921/2021. *Tractatus Logico-Philosophicus* [*TLP*]. Side-by-side ed., K. C. Klement ed., C. K. Ogden, D. F. Pears and B. F. McGuinness trans. [online] Available at: <https://people.umass.edu/~klement/tlp/tlp.pdf> [February 2023].

Wittgenstein, L., 1939/1976. *Wittgenstein's Lectures on the Foundations of Mathematics, Cambridge, 1939: From the Notes of R.G. Bosanquet, Norman Malcolm, Rush Rhees, Yorick Smithies* [*LFM*]. C. Diamond ed., Ithaca, NY: Cornell University Press.

Wittgenstein, L., 1953/2009. *Philosophical Investigations* [*PI*]. 4th ed., G. E. M. Anscombe, P. M. S. Hacker and J. Schulte trans., P. M. S. Hacker and J. Schulte ed., Oxford: Wiley- Blackwell.

Wittgenstein, L., 1956/1978. *Remarks on the Foundations of Mathematics* [*RFM*]. 3rd ed., G. H. von Wright, R. Rhees and G. E. M. Anscombe eds., G. E. M. Anscombe trans., Oxford: Basil Blackwell.

Wittgenstein, L., 1974/1980. *Philosophical Grammar* [*PG*]. R. Rhees ed., A. Kenny trans., Malden, MA: Blackwell Publishing.

Wittgenstein, L., 1979. *Wittgenstein and the Vienna Circle: Conversations Recorded by Friedrich Waismann* [*WWK*]. B. McGuinness ed., J. Schulte and B. McGuinness trans., Oxford: Basil Blackwell.

Biographical Note

Martin Gullvåg Sætre is a PhD candidate at the University of Bergen, where he is a member of the project *Mathematics with a Human Face: Set Theory within a Naturalized Wittgensteinian Framework*. His dissertation, *Wittgenstein on the Anthropological Grounds of Mathematics*, delves into the later Wittgenstein's writings and lectures on mathematics and situates them within Wittgenstein's anthropologically oriented philosophy.